Brownian motion constrained to enclose a given area

1991 J. Phys. A: Math. Gen. 24 L561

(http://iopscience.iop.org/0305-4470/24/10/012)

J. Phys. A: Math. Gen. 24 (1991) L561–L564. Printed in the UK

LETTER TO THE EDITOR

Brownian motion constrained to enclose a given area

A D Verga

Equipe Turbulence Plasma†, Institut Méditerranéen de Technologie, F-13451 Marseille Cedex 13, France

Received 14 January 1991

Abstract. The probability of a planar Brownian closed curve enclosing a fixed area is rederived using a simple method of functional integration. The mean square distance between two points of the ring is calculated. When the constraint on the enclosed area increases, one passes from a diffusion regime of the Brownian ring to a normal regime, where the curve approaches a circle.

Random walks under global constraints are an active field of research which embodies a variety of physical problems. Recently there was a renewal of interest in the so-called stochastic area Lévy problem [1]. A particle performs a random ring in the plane constrained to enclose an area A. This stochastic process was introduced by Brereton and Butler [2] in polymer physics as a simplified model of polymer entanglements. The computation of the probability $P(A)$ to enclose an area and other related quantities, was reconsidered by Khandekar and Wiegel [3], and more recently by Duplantier [4]. The problem is reduced to a Gaussian functional integral which can be computed exactly. In this letter I calculate by a simple method the probability $P(A)$ and I give an exact formula for the mean square distance between two points of the ring. I demonstrate that when the imposed area increases, the ring tends to a circle.

Let $t \in [0, T]$ be a real parameter and $\boldsymbol{r} = \boldsymbol{r}(t)$ the parametric equation of a closed curve in the plane $\mathscr{R}^2$. The algebraic area enclosed by the curve is

$$\mathscr{A}[\boldsymbol{r}] = \frac{1}{2}\int_0^T \mathrm{d}t\, \boldsymbol{r}\varepsilon\dot{\boldsymbol{r}} \tag{1}$$

where the overdot means derivation with respect to the parameter t and ε is the unit antisymmetric matrix. Assume that $\boldsymbol{r}(t)$ is actually a two-dimensional periodic Brownian motion with period T, then $\boldsymbol{r}(T) = \boldsymbol{r}(0)$ is imposed. The statistical properties of the stochastic area process can be deduced from the characteristic function

$$Z(g, \boldsymbol{q}) = \frac{1}{Z_0}\int \mathscr{D}\boldsymbol{r}\, \exp(-\mathrm{i}g\mathscr{A}[\boldsymbol{r}])\exp[\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{r}(t_2) - \boldsymbol{r}(t_1))]$$
$$\times \exp\left(-\frac{1}{2}\int_0^T \mathrm{d}t\, \dot{\boldsymbol{r}}^2\right)\delta(\boldsymbol{r}(T) - \boldsymbol{r}(0)) \tag{2}$$

where t_1 and t_2 are two intermediate values of the curve parameter designating two arbitrary positions on the curve (for simplicity in the notation, the explicit dependence

† Laboratoire des Interactions Ioniques et Moléculaires, Université de Provence, URA 773, Centre National de la Recherche Scientifique.

0305-4470/91/100561+04$03.50

of Z on these parameters is omitted), Z_0 obeys the normalization $Z(0,0)=1$. In formula (2) the δ-function takes into account the constraint of a closed Brownian curve. The probability to enclose an area A is given by

$$P(A)=\int_{-\infty}^{\infty}\frac{\mathrm{d}g}{2\pi}\,\mathrm{e}^{\mathrm{i}gA}Z(g,0). \tag{3}$$

Some information about the form of the Brownian ring can be deduced from the mean square distance between points $\boldsymbol{r}(t_1)$ and $\boldsymbol{r}(t_2)$

$$R^2\equiv\langle|\boldsymbol{r}(t_2)-\boldsymbol{r}(t_1)|^2\rangle=-\frac{\partial^2}{\partial\boldsymbol{q}^2}\ln Z(A,\boldsymbol{q})\bigg|_{\boldsymbol{q}=0} \tag{4}$$

where $Z(A,\boldsymbol{q})$ is the Fourier transform with respect to g of $Z(g,\boldsymbol{q})$.

It is worth noting that the function $Z(g,\boldsymbol{q})$ is a Gaussian functional integral, and hence can be calculated exactly. The simplest way to compute this functional integral is to represent the Brownian path $\boldsymbol{r}(t)$ by a Fourier sum

$$\boldsymbol{r}(t)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\boldsymbol{r}_n\,\mathrm{e}^{2\pi\mathrm{i}nt/T}\qquad(\boldsymbol{r}_n^*=\boldsymbol{r}_{-n}) \tag{5}$$

which automatically satisfies the boundary conditions of a closed curve. Given that (5) is a linear transformation of the path, the integral measure becomes a product measure on the complex vectors $\boldsymbol{r}_n:\mathscr{D}\boldsymbol{r}\to\Pi_{n=1}^{\infty}\,\mathrm{d}^2\boldsymbol{r}_n$, where the constants are absorbed in the normalization factor. This transformation follows from the fact that a Wiener process can be expanded in a countable coordinate system [5]. Using this transformation and standard Gaussian functional integration one obtains, for the generating function $Z(g,\boldsymbol{q})$, the following expression

$$Z(g,\boldsymbol{q})=\frac{x}{\sinh x}\exp\left\{-\frac{Tq^2}{2x\sinh x}\sinh\tau x\sinh[(1-\tau)x]\right\} \tag{6}$$

where $x=gT/2$ and $\tau=(t_2-t_1)/T$. The fact that $Z(g,\boldsymbol{q})$ only depends on the time difference τ reflects the translational invariance of the stochastic area process. Moreover, $Z(g,\boldsymbol{q})$ only depends on the modulus of the vector $\boldsymbol{q}$ expressing the isotropy of the Brownian ring. The Fourier transform with respect to $\boldsymbol{q}$, $Z(g,\boldsymbol{r})$, of the generating function (6) is related to the characteristic function calculated by Duplantier [4].

The probability to enclose an area A is obtained introducing (6) into (3)

$$P(A)=\int_{-\infty}^{\infty}\frac{\mathrm{d}g}{2\pi}\,\mathrm{e}^{\mathrm{i}gA}\frac{gT}{2\sinh(gT/2)}=\frac{\pi}{2T}\operatorname{sech}^2(\pi A/T) \tag{7}$$

which is Lévy's result. From formula (4), using (6) and $P(A)=Z(A,0)$, the mean square distance is expressed as an integral which is easily calculated using the residue theorem. One finally obtains the exact formula for the mean square distance between two arbitrary points of the Brownian ring

$$R^2=\frac{2T}{\pi}\frac{\sin\pi\tau}{\cosh\pi a+\cos 2\pi\tau}\left[\frac{a}{2}\sinh\pi a\sin\pi\tau+(1-2\tau)\cosh^2(\pi a/2)\cos\pi\tau\right]. \tag{8}$$

Figures 1 and 2 show the behaviour of $R_{A,T}(\tau)$ as a function of the parameters T and A. Two regimes of the stochastic area process can be distinguished, depending on the value of the parameter A/T. When A/T is small, the area constraint becomes negligible, and the size of the Brownian ring grows as in a normal diffusion process, that is $R^2\sim T$.

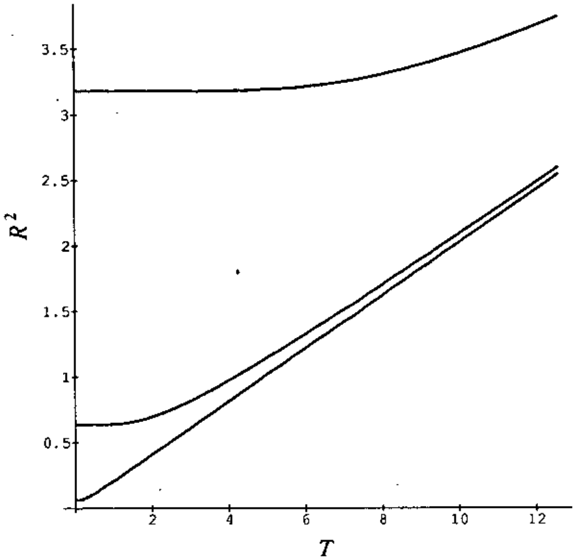

Figure 1. Mean square distance R^2 at $\tau=\frac{1}{2}$, as a function of T, for $A=0.1$ (bottom), 1 (middle) and 10 (top).

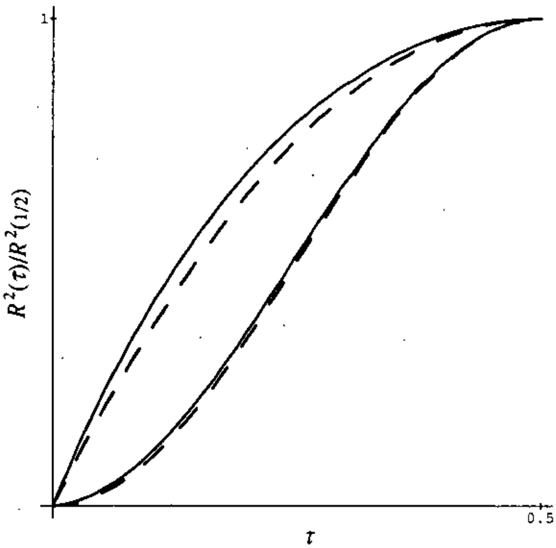

Figure 2. Mean square distance R^2 with $T=1$, normalized at its $\tau=\frac{1}{2}$ value, as a function of τ, for $A=0$ (top solid), 10 (bottom solid). For comparison the free Brownian ring (top dashed) and the circle (bottom dashed) are included.

However, the diffusion coefficient D differs from the pure (unconstrained) Brownian ring value:

$$D=\lim_{T\to\infty}\frac{R^2_{A,T}(\tau)}{2T}=\frac{1-2\tau}{2\pi}\tanh\pi\tau \qquad (A \text{ fixed}). \tag{9}$$

For a pure Brownian ring the diffusion coefficient is $D = \tau(1-\tau)$. In fact, this limit corresponds to the value of R^2 at $A=0$, the most probable enclosed area.

On the other hand, when the ratio A/T is large, R^2 becomes independent of T. Therefore, a Brownian ring enclosing a given area A, considered as a function of T, conserves its size until a time of order A, and then starts to diffuse.

Figure 2 represents the mean square distance R^2 as a function of τ, for different values of the enclosed area. In this figure, which intends to show the shape of the Brownian ring, R^2 is normalized by its value at $\tau=\frac{1}{2}$, and $T=1$. For reference the graph of R^2 for a free Brownian ring (without the area constraint) and the one for a circle, are also shown. It is remarkable that when the area constraint becomes dominant (with respect to the diffusion behaviour) the mean square distance between two points of the Brownian ring approaches the radius of a circle; for a circle (the origin of coordinates is on the circumference), the radius is given by $r^2(t) \sim 1-\cos 2\pi t$ ($t \in [0, 1]$).

To resume the above discussion, let us consider the area as a function of the parameter T, take $A(T) = T^\alpha$. One may distinguish two cases. First, for $\alpha < 1$ the mean square distance behaves as

$$\lim_{T\to\infty} R^2_{T^\alpha, T}(\tau) = 2DT \qquad (\alpha < 1) \tag{10}$$

with D given by (9). Note that the Brownian ring grows diffusively irrespective of the value of α, smaller than one. For $\alpha = 1$, the Brownian ring also expand diffusively but with a different diffusion coefficient. Second, for $\alpha > 1$ one finds the normal scaling $R^2 \sim A$

$$\lim_{T\to\infty} R^2_{T^\alpha, T}(\tau) = \frac{T^\alpha}{\pi}(1-\cos 2\pi\tau) \qquad (\alpha > 1) \tag{11}$$

and the shape of the ring tends to that of a circle; when the area constraint dominates, the random motion approaches a deterministic one.

In conclusion, I have demonstrated that a Brownian ring, constrained to enclose a given area, presents two regimes depending on the relative importance of the global constraint. In the first regime the stochastic area process behaves as a diffusion, when the ratio A/T vanishes for large T. In the second regime, in the opposite case, the ratio A/T increases with T, where the stochastic area process scales normally ($R^2 \sim A$) and the averaged shape of the ring becomes a circle.

I thank F Dunlop for useful discussions.

References

[1] Lévy P 1940 *Am. J. Math.* **62** 427; 1950 *C. R. Acad. Sci., Paris* **230** 430, 689(E); 1951 *Proc. 2nd Berkeley Symp. on Mathematical Statistics and Probability* (Berkeley, CA: University of California Press) p 171
[2] Brereton M G and Butler C 1987 *J. Phys. A: Math. Gen.* **20** 3955
[3] Khandekar D C and Wiegel F W 1988 *J. Phys. A: Math. Gen.* **21** L563; 1989 *J. Physique* **50** 263
[4] Duplantier B 1989 *J. Phys. A: Math. Gen.* **22** 3033
[5] Breiman L 1968 *Probability* (Reading, MA: Addison-Wesley) p 259